Towards a Quantum Theory without 'Quantization'

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1. INTRODUCTION: QUANTUM THEORY VERSUS QUANTIZATION THEORY

Bryce DeWitt is a philosophical realist. He believes that the world exists objectively and that the task of physics is to obtain as true as possible a description of it. He has always contended that we should take seriously the assertions of our theories, to 'push them to their limits' until they either fail or yield new insights into the nature of reality. He has opposed *ad hoc* attempts to reformulate newer, better but uncomfortable theories in terms of the formalism, kinematics and ontology of familiar but obsolete theories. Thus for him, general relativity is a theory of the dynamics of space–time geometry, not just another field theory on a Minkowskian flat space–time background. And quantum theory is an objective theory of parallel interfering universes, not of a sequence of subjective classical experiences.

Quantum theory has been extraordinarily slow in freeing itself from the apron strings of its classical ancestors. In championing and developing Everett's interpretation (DeWitt and Graham 1973) DeWitt has been instrumental in the exorcism of classical concepts from the *interpretation* of quantum theory. But in the more important matter of formalism we still know of no other way of constructing quantum theories than 'quantization', a set of semi-explicit *ad hoc* rules for making a silk purse (a quantum theory) out of a sow's ear (the associated classical theory). And even DeWitt, who originated most of the ideas presented in this article, now appears to acquiesce in this.

I believe that quantization will have to go before further progress is made at the foundations of physics.

Perhaps the reason is best illustrated by analogy: suppose that in an elementary chemistry textbook, in the chapter on combustion, no mention were made of oxygen. Instead, the chapter begins with a detailed exposition of the theory of phlogiston. It then explains that this theory is now known to be false, but that a better theory may be constructed from it by means of 'chemicalization rules': 'phlogiston must be thought of formally as occupying a negative volume', and so forth. These rules are numerous, ramshackle and without independent motivation but (in experienced hands) they do correctly predict the results of experiments. Usually. If chemistry were really in the state indicated by such a textbook, it would bode ill for the future of the subject. Progress would be halted until chemists stopped thinking in terms of phlogiston and someone invented a theory of oxygen.

To base the theory of quantum fields $\hat{\varphi}_i$ on that of classical fields φ_i is like basing chemistry on phlogiston or general relativity on Minkowski space–time: it can be done, up to a point, but it is a mistake; not only because the procedure is ill defined and the resulting theory of doubtful consistency, but because the world isn't really like that. *No classical fields φ_i exist in nature.* Like phlogiston, they were participants in obsolete physical theories. Only when the quantum formalism contains no reference to classical theory can we hope to understand what it says. And only then can we hope to improve upon it.

It might be objected that the Correspondence Principle gives the classical theory of φ_i a special place in the quantum theory of $\hat{\varphi}_i$. But this is just a confusion created by quantization theory. As we shall see, the Correspondence Principle can be formulated without reference to classical theory.

2. QUANTUM THEORY WITHOUT CLASSICAL THEORY

I shall now try to set up the formalism of quantum theory without referring to classical concepts noting, but not always solving, the problems as they arise.

An observable $\hat{\varphi}$ is something whose numerical value can in principle be measured, or could be measured if the requisite measuring apparatus were present at the appropriate time(s) and place(s). The set $\{\varphi\}$ of possible outcomes of a measurement of $\hat{\varphi}$ is called the spectrum of $\hat{\varphi}$. If the spectrum of $\hat{\varphi}$ is independent of time, $\hat{\varphi}$ is said to have 'no explicit time dependence'. In quantum theory, observables $\hat{\varphi}$ are represented by Hermitian operators and the values $\{\varphi\}$ are their eigenvalues.

The observables in nature are quantum fields; that is, parametrized sets of observables $\hat{\varphi}_i$. The index i represents a set of parameters such as space–time coordinates, tensor indices, internal symmetry indices and enumeration indices. At present it is customary always to include a time coordinate t among the parameters. The fact that something with as fundamental a physical significance as the time appears only as a parameter is somewhat unsatisfactory and may be eliminated in future formulations (Page and Wootters 1982) of quantum theory.

The state of the world is represented by a unit vector, the state vector $|\psi\rangle$, in the Hilbert space spanned by the eigenvectors of any maximal commuting set of observables. This is not the place to describe the mechanism by which the formal structure of vectors and Hermitian operators is asserted by quantum theory to correspond to reality. The reader is referred to the work of Everett (DeWitt and Graham 1973; Deutsch 1980). Suffice it to say that the *interpretation* of quantum theory has been shown to require no classical element. When the same has been done for the *formalism*, quantum theory will have come of age.

The dynamics of quantum fields are generated by a principle of stationary action. The action is an *operator*-valued functional

$$\hat{S}[\hat{\varphi}_i] \tag{2.1}$$

of the field observables. Under an infinitesimal variation

$$\hat{\varphi}_i \to \hat{\varphi}_i + \delta\hat{\varphi}_i \tag{2.2}$$

$$\hat{S} \to \hat{S} + \sum_i \frac{\delta \hat{S}}{\delta \hat{\varphi}_i}\, \delta\hat{\varphi}_i. \tag{2.3}$$

An ambiguity of notation arises in (2.3) (which can serve as a formal definition of the functional derivative): it is not clear how the three pairs of implied operator indices are to be connected and summed over. It is necessary to make these indices explicit, writing (2.3) as

$$S_{\hat{\alpha}} \to S_{\hat{\alpha}} + \frac{\delta S_{\hat{\alpha}}}{\delta \varphi_{\hat{\beta}i}}\, \delta\varphi_{\hat{\beta}i} \tag{2.4}$$

where each quantum index $\hat{\alpha}$ stands for a Hermitian pair of Hilbert space labels. I have also adopted the Einstein summation convention both for the quantum indices and for the generalized coordinates i.

The quantum principle of stationary action is not as straightforward as the classical one. It is not in general possible for the full variation $\delta S_{\hat{\alpha}}/\delta\varphi_{\hat{\beta}i}$ to vanish. The indices show that the resulting system of dynamical equations would be over-determined. The correct principle has the form

$$\frac{\delta S_{\hat{\alpha}}[\hat{\varphi}]}{\delta \varphi_{\hat{\beta}i}}\, X_{\hat{\beta}i}{}^{j}[\hat{\varphi}] = 0 \tag{2.5}$$

for some functional $\hat{X}_i{}^j[\hat{\varphi}]$, which is equivalent to the requirement that the action be stationary, not under general variations in $\hat{\varphi}_i$, but only under variations of the form

$$\delta\varphi_{\hat{\alpha}i} = X_{\hat{\alpha}i}{}^{j}\delta\varphi_j \tag{2.6}$$

where the $\delta\varphi_i$ are suitable infinitesimal c-number test functions. It is not known in general how to choose the functional $\hat{X}_i{}^j[\hat{\varphi}]$. Schwinger (1953), Peierls (1952), and DeWitt (1967) all make the natural choice (for Boson fields)

$$\hat{X}_i{}^j = \delta_i{}^j\hat{1} \tag{2.7}$$

which would mean that the action is to be stationary under pure c-number variations in the fundamental field. For certain well studied systems this reproduces the same quantum theory as 'canonical quantization'. But it cannot be the correct choice in general because the variations (2.6) are not in general compatible with the algebra of the operators $\hat{\varphi}_i$. Fermion fields are an obvious example since anticommutators are not invariant under c-number variations. A simple Boson example is the case where i runs from 1 to 3 and $\hat{\varphi}_i$ represents the ith angular momentum component $\hat{L}_i(t)$ of a spin-1 system,

$$[\hat{L}_i(t), \hat{L}_j(t)] = \mathrm{i}\varepsilon_{ij}{}^{k}\hat{L}_k(t). \tag{2.8}$$

Variations $\delta L_i \hat{1}$ are incompatible with (2.8). One compatible choice would be

$$\delta \hat{L}_i = \mathrm{i}\varepsilon_i{}^{jk} \delta L_j(t) \hat{L}_k(t) \tag{2.9}$$

corresponding to

$$X_{\hat{\alpha}i}{}^j[\hat{L}] = \mathrm{i}\varepsilon_i{}^{jk} L_{\hat{\alpha}k}. \tag{2.10}$$

Since different variations $\hat{X}_i{}^j \delta\varphi_j$ generate different stationary action principles and different dynamics, it follows that quantum theory is not covariant under coordinate transformations in configuration space, at least not in any sense known at present. This is in marked contrast to classical theory where the variational principle

$$\frac{\delta S[\varphi]}{\delta \varphi_i} = 0 \tag{2.11}$$

implies

$$\frac{\delta S}{\delta \chi_i} = \frac{\delta S}{\delta \varphi_j} \frac{\delta \varphi_j}{\delta \chi_i} = 0. \tag{2.12}$$

In the quantum case

$$\frac{\delta S_{\hat{\alpha}}}{\delta \chi_{\hat{\beta}i}} X_{\hat{\beta}i}{}^j = \frac{\delta S_{\hat{\alpha}}}{\delta \phi_{\hat{\beta}r}} \frac{\delta \phi_{\hat{\beta}i}}{\delta \chi_{\hat{\gamma}j}} X_{\hat{\gamma}j}{}^k \tag{2.13}$$

which cannot in general be required to vanish whenever $(\delta\hat{S}/\delta\phi_{\hat{\beta}i}) X_{\hat{\beta}i}{}^j$ does.

Coordinate invariance in the base space (i.e. parameter space) can of course still be maintained in quantum theory, but gauge invariance (at least for non-Abelian gauge groups) cannot. What is to become of these cherished invariances of classical field theory—of what property of the quantum theory they are the limiting cases—is an open question. Perhaps the quantum action principle is invariant only under some special class of transformations. Or more interestingly, perhaps the $\hat{X}_i{}^j$ itself suffers changes under coordinate transformations in configuration space, such as to preserve coordinate invariance. This raises the possibility of a more general action principle

$$\frac{\delta S_{\hat{\alpha}}}{\delta \phi_{\hat{\beta}i}} X_{\hat{\beta}i}{}^j + \Gamma^{\hat{\beta}j}{}_{\hat{\alpha}} S_{\hat{\beta}} = 0. \tag{2.14}$$

Could this be regarded as a 'covariant functional derivative' with 'connection coefficients' $\hat{\Gamma}^i$ (DeWitt, private communication)?

3. SOME ELABORATION OF THE PURE QUANTUM THEORY

The dynamical equations (2.5) cannot be solved unless the algebra of the operators $\hat{\varphi}_i$ is given. In quantization theory the algebra is determined by setting the commutator of any two observables equal to i times their classical Poisson bracket (as generalized by Peierls and DeWitt). But, as I shall now show, this will not work in the true quantum theory.

Following Peierls and DeWitt I first construct the theory of small disturbances corresponding to the stationary action principle (2.5). If $\hat{\varphi}_i$ satisfies (2.5) and $\delta\hat{\varphi}_i$ is the infinitesimal disturbance in $\hat{\varphi}_i$ caused by a variation $\delta\hat{S}[\hat{\varphi}]$ in the form of the action then

$$F_{\hat{\alpha}}{}^{j\hat{\beta}i}\delta\varphi_{\hat{\beta}i} = -\frac{\delta\delta S_{\hat{\alpha}}}{\delta\varphi_{\hat{\beta}i}}X_{\hat{\beta}i}{}^{j} \tag{3.1}$$

where

$$F_{\hat{\alpha}}{}^{j\hat{\beta}i}[\hat{\varphi}] = \frac{\delta^2 S_{\hat{\alpha}}[\hat{\varphi}]}{\delta\varphi_{\hat{\beta}i}\delta\varphi_{\hat{\gamma}k}}X_{\hat{\gamma}k}{}^{j} + \frac{\delta S_{\hat{\alpha}}[\hat{\varphi}]}{\delta\varphi_{\hat{\gamma}k}}\frac{\delta X_{\hat{\gamma}k}{}^{j}}{\delta\varphi_{\hat{\beta}i}}. \tag{3.2}$$

(3.1) is the quantum equation of small disturbances. $F_{\hat{\alpha}}{}^{j\hat{\beta}i}$ is a bi-operator which we may abbreviate as $\hat{\hat{F}}$. (I have ignored the possibility that $\hat{X}_i{}^j$ might depend on the form of the action.) Like the dynamical equation itself, equation (3.1) is soluble only given a knowledge of the operator algebra. But in any case a Green's function theory can be based on the equation

$$\hat{\hat{F}}\hat{\hat{G}} = -\hat{\hat{1}} \tag{3.3}$$

i.e.

$$F_{\hat{\alpha}}{}^{j\hat{\gamma}k}G_{\hat{\gamma}k}{}^{\hat{\beta}}{}_{l} = -\delta_{\hat{\alpha}}{}^{\hat{\beta}}\delta_{l}{}^{j}. \tag{3.4}$$

Although the Cauchy problem for operator differential equations is in general more complicated than for scalars (because, for example, operators cannot in general be required to vanish at infinity), this should not affect the Green's function theory because there is presumably no reason why a *variation* $\delta\hat{\varphi}_i$ should not vanish in, say, the remote past. Therefore in particular there should exist a unique retarded Green's function $\hat{\hat{G}}^{-}$ which satisfies equation (3.3). As in the classical theory, the right and left inverses of $\hat{\hat{F}}$ are equal,

$$\hat{\hat{G}}^{-}\hat{\hat{F}} = -\hat{\hat{1}} \tag{3.5}$$

and much of the classical Green's function theory can be carried over to the quantum case just by putting extra hats and indices in appropriate places. For example, defining

$$D^{-}{}_{\hat{A}}\hat{B} = \lim_{\varepsilon\to 0}\delta^{-}_{\varepsilon\hat{A}}\hat{B} \tag{3.6}$$

where $\delta^{-}_{\delta\hat{S}}\hat{B}$ is the retarded disturbance in $\hat{B}$ produced by a variation $\delta\hat{S}$ in the action, we have

$$D^{-}_{\hat{A}}B_{\hat{\alpha}} = \frac{\delta B_{\hat{\alpha}}}{\delta\varphi_{\hat{\beta}j}}G^{-\hat{\gamma}}_{\hat{\beta}j}{}_{k}\frac{\delta A_{\hat{\gamma}}}{\delta\varphi_{\hat{\delta}l}}X_{\hat{\delta}l}{}^{k}. \tag{3.7}$$

There is, however, one important exception: the operator $\hat{\hat{F}}$ in the quantum case is not in general self-adjoint, i.e.

$$F_{\hat{\alpha}}{}^{j\hat{\gamma}k} \not\equiv F^{\hat{\gamma}k}{}_{\hat{\alpha}}{}^{j} \tag{3.8}$$

(where raising and lowering of quantum indices denotes Hermitian conjugation). This has the consequence that the Peierls–Poisson–DeWitt bracket

$$(\hat{A}, \hat{B}) = D_{\hat{A}}^{-}\hat{B} - D_{\hat{B}}^{-}\hat{A} \tag{3.9}$$

based on equation (3.7) does not obey the Jacobi identities and cannot therefore be consistently identified with $-i$ times the commutator as convention (and DeWitt) would have it. Peierls was aware of this problem and suggested that extra terms might be added to his bracket to restore the consistency of his quantization scheme. No one has yet found a way of doing this.

The Peierls–Poisson–DeWitt bracket method of specifying the operator algebra is closely related to the Schwinger variational principle which states that

$$\delta\langle \text{out}|\text{in}\rangle = \mathrm{i}\langle \text{out}|\delta\hat{S}|\text{in}\rangle \tag{3.10}$$

under a variation $\delta\hat{S}$ in the form of the action, where $|\text{out}\rangle$ and $|\text{in}\rangle$ are states corresponding to fixed eigenvalues of observables with no explicit time dependence and confined to the future and past respectively of $\delta\hat{S}$ (it is assumed that $\delta\hat{S}[\hat{\varphi}]$ is constructed from field quantities confined to some parameter space region of effectively finite duration). Using retarded boundary conditions we have

$$\delta|\text{in}\rangle = 0 \tag{3.11}$$

so equation (3.10) implies

$$\delta|\text{out}\rangle = -\mathrm{i}\delta\hat{S}|\text{out}\rangle \tag{3.12}$$

and hence if $\hat{A}[\hat{\varphi}]$ is an 'out' observable,

$$\delta\hat{A} = -\mathrm{i}[\delta\hat{S}, \hat{A}]. \tag{3.13}$$

But $\delta\hat{A}$ can also be represented dynamically *via* the theory of small disturbances

$$\delta\hat{A} = D_{\delta S}^{-}\hat{A}. \tag{3.14}$$

Since the commutator is anti-symmetric in its arguments and $D_{\hat{A}}^{-}\delta S$ vanishes, it follows that

$$[\delta\hat{S}, \hat{A}] = \mathrm{i}(D_{\delta\hat{S}}^{-}\hat{A} - D_{\hat{A}}^{-}\delta\hat{S}) \tag{3.15}$$

which, since $\delta\hat{S}$ and $\hat{A}$ are effectively arbitrary, is the quantum analog of the Peierls expression. Unfortunately, advanced boundary conditions give a different answer

$$[\delta\hat{S}, A] = \mathrm{i}(D_{\hat{A}}^{+}\delta\hat{S} - D_{\delta S}^{+}\hat{A}). \tag{3.16}$$

This signals an inconsistency in the dynamics implied by equation (3.10). The c-number analogs of equation (3.15) and equation (3.16) are indeed identical, but equation (3.8) causes the reciprocity relation between $D_{\hat{A}}^{+}\hat{B}$ and $D_{\hat{B}}^{-}\hat{A}$ to fail in the quantum case. The argument that equations (3.15) and (3.16) differ only by factor-ordering ambiguities seems particularly hollow in this instance, but it does imply that equation (3.15) is true to the lowest order in $\hbar$.

It therefore appears that the Schwinger principle is, as it stands, inconsistent in the full quantum theory. Again it is not known how to modify it.

4. PERTURBATION THEORY AND THE CORRESPONDENCE PRINCIPLE

Fortunately, many of the physical predictions of quantum theory can be obtained without a complete knowledge of the operator algebra. This is the reason why quantization theory, in spite of its cavalier treatment of the 'factor-ordering problem' can have a measure of empirical success. And it is the reason why a c-number theory can exist as a limiting case. The fact that factor-ordering 'ambiguities' are of order $\hbar$ in perturbative expansions of physical quantities is an expression of the Correspondence Principle. To see how this convenient property arises, let us develop the archetypal perturbation method of quantum field theory, the *background field method.*

The objective of background field schemes is always to describe as much as possible of the system in terms of c-number fields. In particular, Schwinger (Schwinger 1953, DeWitt 1965, 1982) found that much can be learned about a quantum theory by investigating the effect of adding a linear source term

$$\hat{S} \rightarrow \hat{S} + J^i \hat{\varphi}_i \tag{4.1}$$

to the action functional. The J^i are c-number 'external sources'. Schwinger's starting point was his variational principle equation (3.10) which I shall assume is true to a sufficiently high order in $\hbar$ to make the following meaningful.

Equation (3.10) implies directly

$$\langle \text{out} | \, T(\hat{A}[\hat{\varphi}]) | \text{in} \rangle = \vec{A} \left[\frac{\delta}{\delta i J} \right] \langle \text{out} | \text{in} \rangle \tag{4.2}$$

where T is the time ordering symbol and $\vec{A}$ is the same functional of the $\delta/\delta i J_i$ as $\hat{A}$ is of the $\hat{\varphi}_i$.

DeWitt introduced in addition to the external source J^i a second c-number field φ_i, a so-called 'classical solution' of the dynamical equations (2.5). $\varphi_i \hat{1}$ will serve as a zeroth approximation to $\hat{\varphi}_i$. Let us consider only the case where $\hat{X}_i{}^j = \hat{1}\delta_i{}^j$. Then the classical solution is defined to satisfy

$$\frac{\delta \hat{S}[\varphi_i \hat{1}]}{\delta \varphi_{\hat{\beta} j}} 1_{\hat{\beta}} = -J_i \hat{1}. \tag{4.3}$$

(For non-trivial $\hat{X}_i{}^j$ one would have to define the 'classical solution' as a q-number such as $\varphi_i Y^i_{\hat{\alpha} j}$, which would raise interesting interpretational problems. This has not been explored to my knowledge.) φ_i is not really a solution of equation (4.3) because $\varphi_i \hat{1}$ will not satisfy any non-trivial commutation relations which, as I have said, must tacitly accompany equation (4.3).

Provided that $\hat{S}$ acquires its operator character solely *via* the $\hat{\varphi}_i$, φ_i will also be the solution of an associated c-number ('classical') variational problem, for the action functional $S[\varphi_i]$ where

$$S_{\hat{\alpha}}[\varphi_i \hat{1}] = S[\varphi_i] 1_{\hat{\alpha}}. \tag{4.4}$$

Writing

$$\hat{\varphi}_i = \varphi_i \hat{1} + \hat{\phi}_i \tag{4.5}$$

we now regard $\hat{\phi}_i$ as formally 'small' in a perturbative scheme for solving equation (2.5). The intuitive justification for this is that $\varphi_i \hat{1}$ differs from the true solution only because all the commutators $[\hat{\varphi}_i, \hat{\varphi}_j]$ have been set to zero in solving the dynamical equations (3.3)—and these commutators ought to be proportional to positive powers of $\hbar$. Nevertheless one can at most hope to represent $\hat{\varphi}_i$ by such a perturbation expansion over some finite part of its (usually) infinite spectrum. For the term $\varphi_i \hat{1}$ will not dominate an unbounded operator, however many powers of $\hbar$ the latter contains. However in regimes where $\hbar$ is 'small', we may continue

$$-J^i 1_{\hat{\alpha}} = \frac{\delta S_{\hat{\alpha}}[\varphi \hat{1}]}{\delta \varphi_{\hat{\beta} i}} 1_{\hat{\beta}} + \frac{\delta^2 S_{\hat{\alpha}}[\varphi \hat{1}]}{\delta \varphi_{\hat{\gamma} j} \delta \varphi_{\hat{\beta} i}} 1_{\hat{\beta}} \hat{\phi}_{\hat{\gamma} j} + \frac{1}{2} \frac{\delta^3 S_{\hat{\alpha}}[\varphi \hat{1}]}{\delta \varphi_{\hat{\delta} k} \delta \varphi_{\hat{\gamma} j} \delta \varphi_{\hat{\beta} i}} 1_{\hat{\beta}} \phi_{\hat{\gamma} j} \hat{\phi}_{\delta k} + \dots . \tag{4.6}$$

Contrary to first appearances, the operator content of the first three functional derivatives in this expansion, given again that $\hat{S}$ depends on no operator independent of the $\hat{\varphi}_i$, is trivial and *independent of the algebra of the* $\hat{\varphi}_i$. This is important for quantization theory since the expansion is useless after the first term containing a factor-ordering ambiguity. In terms of the classical action $S[\varphi]$,

$$-J^i \hat{1} = \frac{\delta \hat{S}[\hat{\varphi}]}{\delta \varphi_{\hat{\beta} i}} 1_{\hat{\beta}} = \frac{\delta^2 S[\varphi]}{\delta \varphi_i \delta \varphi_j} \hat{\phi}_j + \frac{1}{2} \frac{\delta^3 S[\varphi]}{\delta \varphi_i \delta \varphi_j \delta \varphi_k} \hat{\phi}_j \hat{\phi}_k + \dots . \tag{4.7}$$

We are now in a position to derive the centerpiece of quantization theory, the Feynman functional integral formula (DeWitt 1965, 1982), not as an exact theorem but as a remarkable approximation to the true theory. The last term in equation (4.7) can be rewritten as

$$\frac{1}{2} \frac{\delta^3 S[\varphi]}{\delta \varphi_i \delta \varphi_j \delta \varphi_k} T(\hat{\phi}_k \hat{\phi}_j) + \frac{1}{2} T\left(\frac{\delta^3 S[\varphi]}{\delta \varphi_i \delta \varphi_j \delta \varphi_k} \theta(j, k) [\hat{\varphi}_k, \hat{\varphi}_j] \right) + \dots . \tag{4.8}$$

Hence

$$-J^i \hat{1} = T\left(\frac{\delta S[\hat{\varphi}]}{\delta \varphi_i} + \frac{1}{2} \frac{\delta^3 S[\hat{\varphi}]}{\delta \varphi_i \delta \varphi_j \delta \varphi_k} \theta(j, k) [\hat{\varphi}_k, \hat{\varphi}_j] \right) + \dots . \tag{4.9}$$

But since we are supposing that the Peierls expression for the commutator is correct to leading order in $\hbar$,

$$-J^i \hat{1} = T\left(\frac{\delta S[\hat{\varphi}]}{\delta \varphi_i} + \frac{\mathrm{i}}{2} \frac{\delta^3 S[\hat{\varphi}]}{\delta \varphi_i \delta \varphi_j \delta \varphi_k} G^{+kj}[\hat{\varphi}] \right) + \dots \tag{4.10}$$

where G^{+ij} is the *c-number* advanced Green's function. The various functionals of φ_i in (4.9) and (4.10) are promoted to operators by evaluating them with the $\hat{\varphi}_i$s in any order! Finally we have

$$-J^i = T\left(\frac{\delta}{\delta \varphi_i} \left(S[\varphi] + \frac{\mathrm{i}}{2} \ln \det G^+[\varphi] \right) \bigg|_{\varphi \to \hat{\varphi}} \right). \tag{4.11}$$

Still following DeWitt we now express $\langle \text{out} | \text{in} \rangle_{J_i}$ as a functional Fourier transform

$$\langle \text{out} | \text{in} \rangle = \int F[\varphi] \exp(\mathrm{i} J^i \varphi_i) D[\varphi]. \tag{4.12}$$

Functional integration by parts and equations (4.2) and (4.11) give successively

$$\int \left(\frac{\delta}{\delta \varphi_i} F[\varphi] \right) \exp(\mathrm{i} J^j \varphi_j) D[\varphi] = -J^i \langle \text{out} | \text{in} \rangle \tag{4.13}$$

$$= \left\langle \text{out} \left| T \left(\frac{\delta}{\delta \varphi_i} \left(S[\varphi] + \frac{\mathrm{i}}{2} \ln \det G^+[\varphi] \right) \bigg|_{\varphi \to \hat{\varphi}} \right) \right| \text{in} \right\rangle \tag{4.14}$$

$$= \left[\frac{\delta}{\delta \varphi_i} \left(S[\varphi] + \frac{\mathrm{i}}{2} \ln \det G^+[\varphi] \right) \right]_{\varphi_j \to \frac{\delta}{\delta \mathrm{i} J^j}} \langle \text{out} | \text{in} \rangle \tag{4.15}$$

$$= \int \frac{\delta}{\delta \varphi_i} \left(S[\varphi] + \frac{\mathrm{i}}{2} \ln \det G^+[\varphi] \right) F[\varphi] \exp(\mathrm{i} J^j \varphi_j) D[\varphi]. \tag{4.16}$$

It follows that

$$\frac{\delta}{\delta \varphi_i} F[\varphi] = \mathrm{i} \frac{\delta}{\delta \varphi_i} \left(S[\varphi] + \frac{\mathrm{i}}{2} \ln \det G^+[\varphi] \right) F[\varphi] \tag{4.17}$$

$$F[\varphi] \propto \exp(\mathrm{i} S[\varphi]) (\det G^+[\varphi])^{-1/2} \tag{4.18}$$

and

$$\langle \text{out} | \text{in} \rangle \propto \int \exp \mathrm{i} (S[\varphi] + J^j \varphi_j) (\det G^+[\varphi])^{-1/2} D[\varphi]. \tag{4.19}$$

CONCLUSION

We have seen how classical theory arises as a zeroth approximation and quantization theory as a first approximation to real quantum theory. How fortunate it is that we can say so much about the approximations to a theory the true nature of which is still so mysterious.

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